

# DIFFUSION LIMIT FOR THE RADIATIVE TRANSFER EQUATION PERTURBED BY A MARKOVIAN PROCESS

A. DEBUSSCHE<sup>\*</sup>, S. DE MOOR<sup>\*</sup> AND J. VOVELLE<sup>†</sup>

## Abstract

We study the stochastic diffusive limit of a kinetic radiative transfer equation, which is non linear, involving a small parameter and perturbed by a smooth random term. Under an appropriate scaling for the small parameter, using a generalization of the perturbed test-functions method, we show the convergence in law to a stochastic non linear fluid limit.

**Keywords:** Kinetic equations, non-linear, diffusion limit, stochastic partial differential equations, perturbed test functions, Rosseland approximation, radiative transfer.

## 1 Introduction

In this paper, we are interested in the following non-linear equation

$$\begin{cases} \partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L(f^\varepsilon) + \frac{1}{\varepsilon} f^\varepsilon m^\varepsilon, \\ f^\varepsilon(0) = f_0^\varepsilon, \quad t \in [0, T], x \in \mathbb{T}^N, v \in V. \end{cases} \quad (1.1)$$

where  $(V, \mu)$  is a measured space,  $a : V \rightarrow \mathbb{R}^N$ ,  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ . The notation  $\bar{f}$  stands for the average over the velocity space  $V$  of the function  $f$ , that is

$$\bar{f} = \int_V f d\mu(v).$$

The operator  $L$  is a linear operator of relaxation which acts on the velocity variable  $v \in V$  only. It is given by

$$L(f) := \bar{f}F - f, \quad (1.2)$$

where  $v \mapsto F(v)$  is a velocity equilibrium function such that

$$F > 0 \text{ a.s.}, \quad \bar{F} = 1, \quad \sup_{v \in V} F(v) < \infty. \quad (1.3)$$

The term  $m^\varepsilon$  is a random process depending on  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N$  (see section 2.2). The precise description of the problem setting will be given in the next section. In this paper, we study the behaviour in the limit  $\varepsilon \rightarrow 0$  of the solution  $f^\varepsilon$  of (1.1).

Concerning the physical background in the deterministic case ( $m^\varepsilon \equiv 0$ ), equation (1.1) describes the interaction between a surrounding continuous medium and a flux of photons radiating through it in the absence of hydrodynamical motion. The unknown  $f^\varepsilon(t, x, v)$  then stands for a distribution function of photons having position  $x$  and velocity  $v$  at time  $t$ . The

<sup>\*</sup>IRMAR, ENS Rennes, CNRS, UEB. av Robert Schuman, F-35170 Bruz, France. Email: arnaud.debussche@ens-rennes.fr; sylvain.demoor@ens-rennes.fr

<sup>†</sup>Université de Lyon ; CNRS ; Université Lyon 1, Institut Camille Jordan, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France. Email: vovelle@math.univ-lyon1.fr

function  $\sigma$  is the opacity of the matter. When the surrounding medium becomes very large compared to the mean free paths  $\varepsilon$  of photons, the solution  $f^\varepsilon$  to (1.1) is known to behave like  $\rho F$  where  $\rho$  is the solution of the Rosseland equation

$$\partial_t \rho - \operatorname{div}_x (\sigma(\rho)^{-1} K \nabla_x \rho) = 0, \quad (t, x) \in [0, T] \times \mathbb{T}^N,$$

and  $F$  is the velocity equilibrium defined above. This is what we call the Rosseland approximation. In this paper, we investigate such an approximation where we have perturbed the deterministic equation by a smooth multiplicative random noise. To do so, we use the method of perturbed test-functions. This method provides an elegant way of deriving stochastic diffusive limit from random kinetic systems; it was first introduced by Papanicolaou, Stroock and Varadhan [11]. The book of Fouque, Garnier, Papanicolaou and Solna [9] presents many applications to this method. A generalization in infinite dimension of the perturbed test-functions method arose in recent papers of Debussche and Vovelle [7] and de Bouard and Gazeau [6].

In the deterministic case (that is when  $m^\varepsilon \equiv 0$ ), the Rosseland approximation has been widely studied. In the paper of Bardos, Golse and Perthame [1], they derive the Rosseland approximation on a slightly more general equation of radiative transfer type than (1.1) where the solution also depends on the frequency variable  $\nu$ . Using the so-called Hilbert's expansion method, they prove a strong convergence of the solution of the radiative transfer equation to the solution of the Rosseland equation. In [2], the Rosseland approximation is proved in a weaker sense with weakened hypothesis on the various parameters of the radiative transfer equation, in particular on the opacity function  $\sigma$ .

In the stochastic setting, the case where  $\sigma \equiv \sigma_0$  is constant has been studied in the paper of Debussche and Vovelle [7] where they prove the convergence in law of the solution of (1.1) to a limit stochastic fluid equation by mean of a generalization of the perturbed test-functions method. Thus the radiative transfer equation (1.1) is a first step in studying approximation diffusion on non-linear stochastic kinetic equations since the operator  $\sigma(\bar{f})Lf$  stands for a simple non-linear perturbation of the classical linear relaxation operator  $L$ .

As expected, we have to handle some difficulties caused by this non-linearity. In the paper of Debussche and Vovelle [7] is proved the tightness of the family of processes  $(\rho^\varepsilon)_{\varepsilon>0}$  in the space of time-continuous function with values in some negative Sobolev space  $H^{-\eta}(\mathbb{T}^N)$ . In our non-linear setting, this is not any more sufficient to succeed in passing to the limit as  $\varepsilon$  goes to 0. As a consequence, the main step to overcome this difficulty is to prove the tightness of the family of processes  $(\rho^\varepsilon)_{\varepsilon>0}$  in the space  $L^2(0, T; L^2(\mathbb{T}^N))$ . This is made using averaging lemmas in the  $L^2$  setting with a slight adaptation to our stochastic context. The main results about deterministic averaging lemmas that we will use in the sequel can be found in the paper of Jabin [10]. We point out that, thanks to this additional tightness result, we could handle the case of a more general and non-linear noise term in (1.1) of the form  $\frac{1}{\varepsilon} m^\varepsilon \lambda(\bar{f}^\varepsilon) f^\varepsilon$  where  $\lambda : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded and continuous function. In particular, this remains valid in the linear case  $\sigma \equiv 1$  studied in the paper [7] of Debussche and Vovelle so that this paper can provide some improvements to their result.

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## 2 Preliminaries and main result

### 2.1 Notations and hypothesis

Let us now introduce the precise setting of equation (1.1). We work on a finite-time interval  $[0, T]$  where  $T > 0$  and consider periodic boundary conditions for the space variable:  $x \in \mathbb{T}^N$  where  $\mathbb{T}^N$  is the  $N$ -dimensional torus. Regarding the velocity space  $V$ , we assume that  $(V, \mu)$  is a measured space.

In the sequel,  $L^2_{F^{-1}}$  denotes the  $F^{-1}$  weighted  $L^2(\mathbb{T}^N \times V)$  space equipped with the norm

$$\|f\|^2 := \int_{\mathbb{T}^N} \int_V \frac{|f(x, v)|^2}{F(v)} d\mu(v) dx.$$

We denote its scalar product by  $(\cdot, \cdot)$ . We also need to work in the space  $L^2(\mathbb{T}^N)$ , which will be often written  $L^2$  for short when the context is clear. In what follows, we will often use the inequality

$$\|\bar{f}\|_{L^2_x} \leq \|f\|,$$

which is just Cauchy-Schwarz inequality and the fact that  $\bar{F} = 1$ . We also introduce the Sobolev spaces on the torus  $H^\gamma(\mathbb{T}^N)$ , or  $H^\gamma$  for short. For  $\gamma \in \mathbb{N}$ , they consist of periodic functions which are in  $L^2(\mathbb{T}^N)$  as well as their derivatives up to order  $\gamma$ . For general  $\gamma \geq 0$ , they are easily defined by Fourier series. For  $\gamma < 0$ ,  $H^\gamma(\mathbb{T}^N)$  is the dual of  $H^{-\gamma}(\mathbb{T}^N)$ .

Concerning the velocity mapping  $a : V \rightarrow \mathbb{R}^N$ , we shall assume that it is bounded, that is

$$\sup_{v \in V} |a(v)| < \infty. \quad (2.1)$$

Furthermore, we suppose that the following null flux hypothesis holds

$$\int_V a(v) F(v) d\mu(v) = 0, \quad (2.2)$$

and that the following matrix

$$K := \int_V a(v) \otimes a(v) F(v) d\mu(v)$$

is definite positive. Finally, to obtain some compactness in the space variable by means of averaging lemmas, we also assume the following standard condition:

$$\forall \varepsilon > 0, \forall (\xi, \alpha) \in S^{N-1} \times \mathbb{R}, \quad \mu(\{v \in V, |a(v) \cdot \xi + \alpha| < \varepsilon\}) \leq \varepsilon^\theta, \quad (2.3)$$

for some  $\theta \in (0, 1]$ .

Let us now give several hypothesis on the opacity function  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ . We assume that

(H1) There exist two positive constants  $\sigma_*, \sigma^* > 0$  such that for almost all  $x \in \mathbb{R}$ , we have

$$\sigma_* \leq \sigma(x) \leq \sigma^*;$$

(H2) the function  $\sigma$  is Lipschitz continuous.

Similarly as in the deterministic case, we expect with (1.1) that  $\sigma(\overline{f^\varepsilon})L(f^\varepsilon)$  tends to zero with  $\varepsilon$ , so that we should determine the equilibrium of the operator  $\sigma(\overline{\cdot})L(\cdot)$ . In this case, since  $\sigma > 0$ , they are clearly constituted by the functions of the form  $\rho F$  with  $\rho$  being independent of  $v \in V$ . Note that it can easily be seen that  $\sigma(\overline{\cdot})L(\cdot)$  is a bounded operator from  $L_{F-1}^2$  to  $L_{F-1}^2$  and that it is dissipative; precisely, for  $f \in L_{F-1}^2$ ,

$$(\sigma(\overline{f})L f, f) = -\|\sigma^{\frac{1}{2}}(\overline{f})L f\|^2 \leq 0. \quad (2.4)$$

In the sequel, we denote by  $g(t, \cdot)$  the semi-group generated by the operator  $\sigma(\overline{\cdot})L(\cdot)$  on  $L_{F-1}^2$ . It verifies, for  $f \in L_{F-1}^2$ ,

$$\begin{cases} \frac{d}{dt}g(t, f) = \sigma(\overline{g(t, f)})Lg(t, f), \\ g(0, f) = f, \end{cases}$$

and we can show that it is given by

$$g(t, f) = \overline{f}F + (f - \overline{f}F)e^{-t\sigma(\overline{f})}, \quad t \geq 0, f \in L_{F-1}^2.$$

With the hypothesis (H1) made on  $\sigma$ , we deduce the following relaxation property of the operator  $\sigma(\overline{\cdot})L(\cdot)$

$$g(t, f) \longrightarrow \overline{f}F, \quad t \rightarrow \infty, \quad \text{in } L_{F-1}^2. \quad (2.5)$$

## 2.2 The random perturbation

The random term  $m^\varepsilon$  is defined by

$$m^\varepsilon(t, x) := m\left(\frac{t}{\varepsilon^2}, x\right),$$

where  $m$  is a stationary process on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and is adapted to a filtration  $(\mathcal{F}_t)_{t \geq 0}$ . Note that  $m^\varepsilon$  is adapted to the filtration  $(\mathcal{F}_t^\varepsilon)_{t \geq 0} = (\mathcal{F}_{\varepsilon^{-2}t})_{t \geq 0}$ .

We assume that, considered as a random process with values in a space of spatially dependent functions,  $m$  is a stationary homogeneous Markov process taking values in a subset  $E$  of  $W^{1,\infty}(\mathbb{T}^N)$ . In the sequel,  $E$  will be endowed with the norm  $\|\cdot\|_\infty$  of  $L^\infty(\mathbb{T}^N)$ . Besides, we denote by  $\mathcal{B}(E)$  the set of bounded functions from  $E$  to  $\mathbb{R}$  endowed with the norm  $\|g\|_\infty := \sup_{n \in E} |g(n)|$  for  $g \in \mathcal{B}(E)$ .

We assume that  $m$  is stochastically continuous. Note that  $m$  is supposed not to depend on the variable  $v$ . For all  $t \geq 0$ , the law  $\nu$  of  $m_t$  is supposed to be centered

$$\mathbb{E}m_t = \int_E n \, d\nu(n) = 0.$$

We denote by  $e^{tM}$  a transition semi-group on  $E$  associated to  $m$  and by  $M$  its infinitesimal generator.  $D(M)$  stands for the domain of  $M$ ; it is defined as follows:

$$D(M) := \left\{ u \in \mathcal{B}(E), \lim_{h \rightarrow 0} \frac{e^{hM} - I}{h} u \text{ exists in } \mathcal{B}(E) \right\},$$

and if  $u \in D(M)$ , we have

$$Mu := \lim_{h \rightarrow 0} \frac{e^{hM} - I}{h} u \text{ in } \mathcal{B}(E).$$

Moreover, we suppose that  $m$  is ergodic and satisfies some mixing properties in the sense that there exists a subspace  $\mathcal{P}_M$  of  $\mathcal{B}(E)$  such that for any  $g \in \mathcal{P}_M$ , the Poisson equation

$$M\psi = g - \int_E g(n) d\nu(n) =: \widehat{g},$$

has a unique solution  $\psi \in D(M)$  satisfying  $\int_E \psi(n) d\nu(n) = 0$ . We denote by  $M^{-1}\widehat{g}$  this unique solution, and assume that it is given by

$$M^{-1}\widehat{g}(n) = - \int_0^\infty e^{tM} \widehat{g}(n) dt, \quad n \in E. \quad (2.6)$$

In particular, we suppose that the above integral is well defined. We need that  $\mathcal{P}_M$  contains sufficiently many functions. Thus we assume that for all  $f, g \in L^2_{F^{-1}}$ , we have

$$\psi_{f,g}^{(1)} : n \mapsto (fn, g) \in \mathcal{P}_M, \quad (2.7)$$

and we then define  $M^{-1}I$  from  $E$  into  $W^{1,\infty}(\mathbb{T}^N)$  by

$$(fM^{-1}I(n), g) := M^{-1}\psi_{f,g}^{(1)}(n), \quad \forall f, g \in L^2_{F^{-1}}. \quad (2.8)$$

Then, we also suppose that for all  $f, g, h \in L^2_{F^{-1}}$  and all continuous operator  $B$  from  $L^2_{F^{-1}}$  to the space of the continuous bilinear operators on  $L^2_{F^{-1}} \times L^2_{F^{-1}}$ ,

$$\psi_{f,g}^{(2)} : n \mapsto (fnM^{-1}I(n), g), \quad \psi_{B,f,g,h}^{(3)} : n \mapsto B(f)(gn, hM^{-1}I(n)) \in \mathcal{P}_M. \quad (2.9)$$

We need a uniform bound in  $W^{1,\infty}(\mathbb{T}^N)$  of all the functions of the variable  $n \in E$  introduced above. Namely, we assume, for all  $f, g \in L^2_{F^{-1}}$  and all continuous operator  $B$  on  $L^2_{F^{-1}}$ ,

$$\begin{aligned} \|n\|_{W^{1,\infty}(\mathbb{T}^N)} &\leq C_*, & \|M^{-1}I(n)\|_{W^{1,\infty}(\mathbb{T}^N)} &\leq C_*, \\ |M^{-1}\psi_{f,g}^{(2)}| &\leq C_* \|f\| \|g\|, & |M^{-1}\psi_{B,f,g}^{(3)}| &\leq C_* \|B(f)\| \|f\| \|g\|. \end{aligned} \quad (2.10)$$

Finally, we suppose that for all  $f, g \in L^2_{F^{-1}}$ ,

$$n \mapsto (fM^{-1}I(n), g)^2 \in D(M) \text{ with } |M[(fM^{-1}I(n), g)^2]| \leq C_* \|f\|^2 \|g\|^2. \quad (2.11)$$

To describe the limiting stochastic partial differential equation, we then set

$$k(x, y) = \mathbb{E} \int_{\mathbb{R}} m_0(y) m_t(x) dt, \quad x, y \in \mathbb{T}^N.$$

We can easily show that the kernel  $k$  belong to  $L^\infty(\mathbb{T}^N \times \mathbb{T}^N)$  and,  $m$  being stationary, that it is symmetric (see [7]). As a result, we introduce the operator  $Q$  on  $L^2(\mathbb{T}^N)$  associated to the kernel  $k$

$$Qf(x) = \int_{\mathbb{T}^N} k(x, y) f(y) dy,$$

which is self-adjoint, compact and non-negative (see [7]). As a consequence, we can define the square root  $Q^{\frac{1}{2}}$  which is Hilbert-Schmidt on  $L^2(\mathbb{T}^N)$ .

**Remark** The above assumptions on the process  $m$  are verified, for instance, when  $m$  is a Poisson process taking values in a bounded subset  $E$  of  $W^{1,\infty}(\mathbb{T}^N)$ .

### 2.3 Resolution of the kinetic equation

In this section, we solve the linear evolution problem (1.1) thanks to a semi-group approach. We thus introduce the linear operator  $A := a(v) \cdot \nabla_x$  on  $L^2_{F^{-1}}$  with domain

$$D(A) := \{f \in L^2_{F^{-1}}, \nabla_x f \in L^2_{F^{-1}}\}.$$

The operator  $A$  has dense domain and, since it is skew-adjoint, it is  $m$ -dissipative. Consequently  $A$  generates a contraction semigroup  $(\mathcal{T}(t))_{t \geq 0}$  (see [4]). We recall that  $D(A)$  is endowed with the norm  $\|\cdot\|_{D(A)} := \|\cdot\| + \|A \cdot\|$ , and that it is a Banach space.

**Proposition 2.1.** *Let  $T > 0$  and  $f_0^\varepsilon \in L^2_{F^{-1}}$ . Then there exists a unique mild solution of (1.1) on  $[0, T]$  in  $L^\infty(\Omega)$ , that is there exists a unique  $f^\varepsilon \in L^\infty(\Omega, C([0, T], L^2_{F^{-1}}))$  such that  $\mathbb{P}$ -a.s.*

$$f_t^\varepsilon = \mathcal{T}\left(\frac{t}{\varepsilon}\right) f_0^\varepsilon + \int_0^t \mathcal{T}\left(\frac{t-s}{\varepsilon}\right) \left( \frac{1}{\varepsilon^2} \sigma(\overline{f_s^\varepsilon}) L f_s^\varepsilon + \frac{1}{\varepsilon} m_s^\varepsilon f_s^\varepsilon \right) ds, \quad t \in [0, T].$$

Assume further that  $f_0^\varepsilon \in D(A)$ , then there exists a unique strong solution  $f^\varepsilon$  which belongs to the spaces  $L^\infty(\Omega, C^1([0, T], L^2_{F^{-1}}))$  and  $L^\infty(\Omega, C([0, T], D(A)))$  of (1.1).

*Proof.* Subsections 4.3.1 and 4.3.3 in [4] gives that  $\mathbb{P}$ -a.s. there exists a unique mild solution  $f^\varepsilon \in C([0, T], L^2_{F^{-1}})$  and it is not difficult to slightly modify the proof to obtain that in fact  $f^\varepsilon \in L^\infty(\Omega, C([0, T], L^2_{F^{-1}}))$  (we intensively use that for all  $t \geq 0$  and  $\varepsilon > 0$ ,  $\|m_t^\varepsilon\|_{W^{1,\infty}(\mathbb{T}^N)} \leq C_*$ ).

Similarly, subsections 4.3.1 and 4.3.3 in [4] gives us  $\mathbb{P}$ -a.s. a strong solution  $f^\varepsilon$  in the spaces  $C^1([0, T], L^2_{F^{-1}})$  and  $C([0, T], D(A))$  of (1.1) and once again one can easily get that in fact  $f^\varepsilon$  belongs to the spaces  $L^\infty(\Omega, C^1([0, T], L^2_{F^{-1}}))$  and  $L^\infty(\Omega, C([0, T], D(A)))$ .  $\square$

**Remark** If  $f_0^\varepsilon \in D(A)$ , we thus have, for  $\varepsilon > 0$  fixed,

$$\sup_{t \in [0, T]} \|f_t^\varepsilon\| + \sup_{t \in [0, T]} \|A f_t^\varepsilon\| \in L^\infty(\Omega). \quad (2.12)$$

### 2.4 Main result

We are now ready to state our main result.

**Theorem 2.2.** *Assume that  $(f_0^\varepsilon)_{\varepsilon > 0}$  is bounded in  $L^2_{F^{-1}}$  and that*

$$\rho_0^\varepsilon := \int_V f_0^\varepsilon d\mu(v) \xrightarrow{\varepsilon \rightarrow 0} \rho_0 \text{ in } L^2(\mathbb{T}^N).$$

*Then, for all  $\eta > 0$  and  $T > 0$ ,  $\rho^\varepsilon := \overline{f^\varepsilon}$  converges in law in  $C([0, T], H^{-\eta}(\mathbb{T}^N))$  and  $L^2(0, T; L^2(\mathbb{T}^N))$  to the solution  $\rho$  to the non-linear stochastic diffusion equation*

$$d\rho - \operatorname{div}_x(\sigma(\rho)^{-1} K \nabla_x \rho) dt = H\rho dt + \rho Q^{\frac{1}{2}} dW_t, \text{ in } [0, T] \times \mathbb{T}^N, \quad (2.13)$$

*with initial condition  $\rho(0) = \rho_0$  in  $L^2(\mathbb{T}^N)$ , and where  $W$  is a cylindrical Wiener process on  $L^2(\mathbb{T}^N)$ ,*

$$K := \int_V a(v) \otimes a(v) F(v) d\mu(v) \quad (2.14)$$

and

$$H := \int_E n M^{-1} I(n) d\nu(n) \in W^{1,\infty}. \quad (2.15)$$

**Remark** The limit equation (2.13) can also be written in Stratonovich form

$$d\rho - \operatorname{div}_x(\sigma(\rho)^{-1} K \nabla_x \rho) dt = \rho \circ Q^{\frac{1}{2}} dW_t.$$

**Notation** In the sequel, we denote by  $\lesssim$  the inequalities which are valid up to constants of the problem, namely  $C_*$ ,  $N$ ,  $\sup_{\varepsilon>0} \|f_0^\varepsilon\|$ ,  $\sup_{v \in V} |a(v)|$ ,  $\sup_{v \in V} F(v)$ ,  $\sigma_*$ ,  $\sigma^*$ ,  $\|\sigma\|_{\text{Lip}}$  and real constants.

### 3 The generator

The process  $f^\varepsilon$  is not Markov (indeed, by (1.1), we need  $m^\varepsilon$  to know the increments of  $f^\varepsilon$ ) but the couple  $(f^\varepsilon, m^\varepsilon)$  is. From now on, we denote by  $\mathcal{L}^\varepsilon$  its infinitesimal generator, that is

$$\mathcal{L}^\varepsilon \varphi(f, n) := \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} [\varphi(f_h^\varepsilon, m_h^\varepsilon) - \varphi(f, n) | (f_0^\varepsilon, m_0^\varepsilon) = (f, n)],$$

where  $\varphi : L_{F^{-1}}^2 \times E \rightarrow \mathbb{R}$  belongs to the domain of  $\mathcal{L}^\varepsilon$ . Thus we begin this section by introducing a special set of functions which lie in the domain of  $\mathcal{L}^\varepsilon$  and satisfy the associated martingale problem.

In the following, if  $\varphi : L_{F^{-1}}^2 \rightarrow \mathbb{R}$  is differentiable with respect to  $f \in L_{F^{-1}}^2$ , we denote by  $D\varphi(f)$  its differential at a point  $f$  and we identify the differential with the gradient.

**Definition 3.1.** We say that  $\varphi : L_{F^{-1}}^2 \times E \rightarrow \mathbb{R}$  is a good test function if

- (i)  $(f, n) \mapsto \varphi(f, n)$  is differentiable with respect to  $f$ ;
- (ii)  $(f, n) \mapsto D\varphi(f, n)$  is continuous from  $L_{F^{-1}}^2 \times E$  to  $L_{F^{-1}}^2$  and maps bounded sets onto bounded sets;
- (iii) for any  $f \in L_{F^{-1}}^2$ ,  $\varphi(f, \cdot) \in D_M$ ;
- (iv)  $(f, n) \mapsto M\varphi(f, n)$  is continuous from  $L_{F^{-1}}^2 \times E$  to  $\mathbb{R}$  and maps bounded sets onto bounded sets.

**Proposition 3.1.** Let  $\varphi$  be a good test function. Then, for all  $(f, n) \in D(A) \times E$ ,

$$\mathcal{L}^\varepsilon \varphi(f, n) = -\frac{1}{\varepsilon} (Af, D\varphi(f)) + \frac{1}{\varepsilon^2} (\sigma(\bar{f})L f, D\varphi(f)) + \frac{1}{\varepsilon} (fn, D\varphi(f)) + \frac{1}{\varepsilon^2} M\varphi(f, n).$$

Furthermore, if  $f_0^\varepsilon \in D(A)$ ,

$$M_\varphi^\varepsilon(t) := \varphi(f_t^\varepsilon, m_t^\varepsilon) - \varphi(f_0^\varepsilon, m_0^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon \varphi(f_s^\varepsilon, m_s^\varepsilon) ds$$

is a continuous and integrable  $(\mathcal{F}_t^\varepsilon)_{t \geq 0}$  martingale, and if  $|\varphi|^2$  is a good test function, its quadratic variation is given by

$$\langle M_\varphi^\varepsilon \rangle_t = \int_0^t (\mathcal{L}^\varepsilon |\varphi|^2 - 2\varphi \mathcal{L}^\varepsilon \varphi)(f_s^\varepsilon, m_s^\varepsilon) ds.$$

*Proof.* We compute the expression of the infinitesimal generator as follows :

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi(f, n) &= \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} [\varphi(f_h^\varepsilon, m_h^\varepsilon) - \varphi(f, n) | (f_0^\varepsilon, m_0^\varepsilon) = (f, n)] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} [\varphi(f_h^\varepsilon, m_h^\varepsilon) - \varphi(f, m_h^\varepsilon) | (f_0^\varepsilon, m_0^\varepsilon) = (f, n)] \\ &\quad + \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} [\varphi(f, m_h^\varepsilon) - \varphi(f, n) | m_0^\varepsilon = n]\end{aligned}$$

Since  $\varphi$  verifies point (iii) of Definition 3.1, the second term of the last equality goes to  $\varepsilon^{-2} M\varphi(f, n)$  when  $h \rightarrow 0$ . We now focus on the first term. With points (i) – (ii) of Definition 3.1, we have that  $\varphi$  is continuously differentiable with respect to  $f$ . Thus

$$\varphi(f_h^\varepsilon, m_h^\varepsilon) - \varphi(f, m_h^\varepsilon) = \int_0^1 D\varphi(f + s(f_h^\varepsilon - f), m_h^\varepsilon)(f_h^\varepsilon - f) ds.$$

Besides, since  $f_0^\varepsilon = f \in D(A)$ ,  $f^\varepsilon \in C^1([0, T], L_{F-1}^2)$  and we have

$$f_h^\varepsilon - f = h \int_0^1 \partial_t f_{uh}^\varepsilon du.$$

Thus, we can rewrite the first term as

$$\begin{aligned}&= \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} [\varphi(f_h^\varepsilon, m_h^\varepsilon) - \varphi(f, m_h^\varepsilon) | (f_0^\varepsilon, m_0^\varepsilon) = (f, n)] \\ &= \lim_{h \rightarrow 0} \mathbb{E}_{(f, n)} \left[ \int_0^1 \int_0^1 a_h(w, s, u) du ds \right],\end{aligned}$$

with  $a_h(w, s, u) := D\varphi(f + s(f_h^\varepsilon - f), m_h^\varepsilon)(\partial_t f_{uh}^\varepsilon)$  and where  $\mathbb{E}_{(f, n)}$  denotes the expectation under the probability measure  $\mathbb{P}_{(f, n)} := \mathbb{P}(\cdot | (f_0^\varepsilon, m_0^\varepsilon) = (f, n))$ .

Recall that  $D\varphi$  is continuous with respect to  $(f, n)$  thanks to point (ii) of Definition 3.1, that  $f^\varepsilon$  is  $\mathbb{P}$ -a.s. in  $C^1([0, T], L_{F-1}^2)$  and that  $m^\varepsilon$  is stochastically continuous to conclude that  $a_h$  converges in probability as  $h \rightarrow 0$  to  $D\varphi(f, n)(\partial_t f^\varepsilon(0))$  in the probability space  $\tilde{\Omega} := (\Omega \times [0, 1] \times [0, 1], \mathbb{P}_{(f, n)} \otimes dx \otimes ds)$ . Furthermore, we prove that  $(a_h)_{0 \leq h \leq 1}$  is uniformly integrable in  $\tilde{\Omega}$  since it is uniformly bounded with respect to  $0 \leq h \leq 1$  in  $L^\infty(\tilde{\Omega})$ . Indeed, with the fact that  $L$  is a bounded operator, with (H1) and the fact that  $\|n\|_{L^\infty(\mathbb{T}^N)} \lesssim 1$  for all  $n \in E$ , we get

$$|a_h| \lesssim \|D\varphi(f + s(f_h^\varepsilon - f), m_h^\varepsilon)\|(\|f_{uh}^\varepsilon\| + \|Af_{uh}^\varepsilon\|).$$

With (2.12), we set

$$R := \sup_{t \in [0, T]} \|f_t^\varepsilon\| + \sup_{t \in [0, T]} \|Af_t^\varepsilon\| \in L^\infty(\Omega),$$

and define  $r := \|R\|_{L^\infty(\Omega)}$ . Then, since  $D\varphi$  maps bounded sets on bounded sets, we can bound the term  $\|D\varphi(f + s(f_h^\varepsilon - f), m_h^\varepsilon)\|$  by

$$C := \sup \left\{ \|D\varphi(f, n)\|, f \in B_{L_{F-1}^2}(0, \|f\| + r), n \in B_E(0, C_*) \right\}.$$

So we are led to

$$\|a_h\|_{L^\infty(\tilde{\Omega})} \lesssim C \cdot r,$$

which is what we announced. To prove the sequel of the proposition, we use the same kind of ideas and follow the proofs of [7, Proposition 6] and [9, Appendix 6.9].  $\square$



## 4 The limit generator

In this section, we study the limit of the generator  $\mathcal{L}^\varepsilon$  when  $\varepsilon \rightarrow 0$ . The limit generator  $\mathcal{L}$  will characterize the limit stochastic fluid equation.

### 4.1 Formal derivation of the corrections

To derive the diffusive limiting equation, one has to study the limit as  $\varepsilon$  goes to 0 of quantities of the form  $\mathcal{L}^\varepsilon \varphi$  where  $\varphi$  is a good test function. To do so, following the perturbed test-functions method, we have to correct  $\varphi$  so as to obtain a non-singular limit. We search the correction  $\varphi^\varepsilon$  of  $\varphi$  under the classical form:

$$\varphi^\varepsilon := \varphi + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2.$$

In this decomposition,  $\varphi_1$  and  $\varphi_2$  are respectively the first and second order corrections and are to be defined in the sequel so that

$$\mathcal{L}^\varepsilon \varphi^\varepsilon = \mathcal{L} \varphi + O(\varepsilon),$$

where  $\mathcal{L}$  will be the limit generator. We restrict our study to smooth test-functions. Precisely, we introduce the set of spatial derivative operators up to order 3:

$$\mathcal{R} := \{\partial_{i_1}^{e_1} \partial_{i_2}^{e_2} \partial_{i_3}^{e_3}, e \in \{0, 1\}^3, i \in \{1, \dots, N\}^3, |i| \leq 3\}$$

and we suppose that the test-function  $\varphi$  is a good test, that  $\varphi \in C^3(L_{F-1}^2)$  and that there exists a constant  $C_\varphi > 0$  such that

$$\begin{cases} |\varphi(f)| \leq C_\varphi(1 + \|f\|^2), \\ \|\Lambda D\varphi(f)\| \leq C_\varphi(1 + \|f\|), \\ |D^2\varphi(f)(\Lambda_1 h, \Lambda_2 k)| \leq C_\varphi \|h\| \|k\|, \\ |D^3\varphi(f)(\Lambda_1 h, \Lambda_2 k, \Lambda_3 l)| \leq C_\varphi \|h\| \|k\| \|l\|, \end{cases} \quad (4.1)$$

for any  $f, h, k, l \in L_{F-1}^2$  and  $\Lambda, \Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{R}$ . Thanks to Proposition 3.1, and since  $\varphi$  does not depend on  $n \in E$ , we can write

$$\mathcal{L}^\varepsilon \varphi^\varepsilon(f, n) = -\frac{1}{\varepsilon}(Af, D\varphi(f)) + \frac{1}{\varepsilon^2}(\sigma(\bar{f})Lf, D\varphi(f)) + \frac{1}{\varepsilon}(fn, D\varphi(f)) \quad (4.2)$$

$$- (Af, D\varphi_1(f)) + \frac{1}{\varepsilon}(\sigma(\bar{f})Lf, D\varphi_1(f)) + (fn, D\varphi_1(f)) + \frac{1}{\varepsilon}M\varphi_1 \quad (4.3)$$

$$- \varepsilon(Af, D\varphi_2(f)) + (\sigma(\bar{f})Lf, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)) + M\varphi_2. \quad (4.4)$$

In the sequel, we do not care about the terms relative to the transport part  $A$  of the equation since these terms will be handled as in the deterministic case (when  $m^\varepsilon \equiv 0$ ). To be more precise, and as it will be shown in the sequel, the first term of (4.2) will give rise, as  $\varepsilon$  goes to 0, to the deterministic term in the limit generator  $\mathcal{L}$  and the first terms of (4.3) and (4.4) are respectively of orders  $\varepsilon$  and  $\varepsilon^2$ . For the remaining terms, in a first step, we would like to cancel those who have a singular power of  $\varepsilon$ . Thus we should impose that the two following equations hold:

$$(\sigma(\bar{f})Lf, D\varphi(f)) = 0, \quad (4.5)$$

$$(\sigma(\bar{f})Lf, D\varphi_1(f)) + M\varphi_1 + (fn, D\varphi(f)) = 0. \quad (4.6)$$

Let us say a word about the fact that we chose to handle the terms relative to the transport part of the equation separately. When trying to correct these terms thanks to the correctors  $\varphi_1$  and  $\varphi_2$ , the non-linearity  $\sigma$  implies that the second corrector  $\varphi_2$ , unless we can write it formally, does not behave properly any more.

#### 4.1.1 Equation on $\varphi$

Let us solve (4.5). We recall that  $(g(t, f))_{t \geq 0}$  denotes the semigroup of the operator  $\sigma(\bar{\cdot})L$ . Equation (4.5) gives immediately that the map  $t \mapsto \varphi(g(t, f))$  is constant. As a result, with (2.5),

$$\varphi(f) = \varphi(g(0, f)) = \varphi(\varphi(g(\infty, f))) = \varphi(\bar{f}F),$$

so that  $\varphi$  only depends on  $\bar{f}F$ . This implies, for all  $h \in L^2_{F^{-1}}$ ,

$$(h, D\varphi(f)) = (\bar{h}F, D\varphi(\bar{f}F)). \quad (4.7)$$

#### 4.1.2 Equation on $\varphi_1$

Next, we solve (4.6). We consider the Markov process  $(g(t, f), m(t, n))_{t \geq 0}$ . Its generator will be denoted by  $\mathcal{M}$ . We observe that equation (4.6) rewrites:

$$\mathcal{M}\varphi_1(f, n) = -(fn, D\varphi(f)).$$

This Poisson equation will have a solution if the integral of  $(f, n) \mapsto (fn, D\varphi(f))$  over  $L^2_{F^{-1}} \times E$  equipped with the invariant measure of the process  $(g(t, f), m(t, n))_{t \geq 0}$  is zero. So, we must verify that

$$\int_E (\bar{f}Fn, D\varphi(\bar{f}F)) d\nu(n) = 0,$$

and this relation does hold since  $m$  is centered. As a consequence, if we can prove the existence of the integral, we can write  $\varphi_1$  as

$$\varphi_1(f, n) = \int_0^\infty \mathbb{E}(g(t, f)m(t, n), D\varphi(g(t, f))) dt.$$

Then, we use (4.7),  $\overline{g(t, f)} = \bar{f}$  and (2.7) and (2.8) to obtain

$$\begin{aligned} \varphi_1(f, n) &= \int_0^\infty \mathbb{E}(\bar{f}Fm(t, n), D\varphi(\bar{f}F)) dt = -(\bar{f}FM^{-1}I(n), D\varphi(\bar{f}F)) \\ &= -(fM^{-1}I(n), D\varphi(f)). \end{aligned}$$

We are now able to state the

**Proposition 4.1** (First corrector). *Let  $\varphi \in C^3(L^2_{F^{-1}})$  be a good test-function satisfying (4.1) and depending only on  $\bar{f}F$ . For any  $(f, n) \in L^2_{F^{-1}} \times E$ , we define the first corrector  $\varphi_1$  as*

$$\varphi_1(f, n) := -(fM^{-1}I(n), D\varphi(f)).$$

*Furthermore, it satisfies the bounds*

$$(i) \quad |\varphi_1(f, n)| \lesssim C_\varphi(1 + \|f\|)^2, \quad (ii) \quad \|AD\varphi_1(f, n)\| \lesssim C_\varphi(1 + \|f\|). \quad (4.8)$$

Note that the bounds (4.8) are consequences of (2.10) and (4.1).

### 4.1.3 Equation on $\varphi_2$

At this stage, we have

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f, n) &= -\frac{1}{\varepsilon}(Af, D\varphi(f)) + \mathcal{M}\varphi_2 + (fn, D\varphi_1(f)) \\ &\quad - (Af, D\varphi_1(f)) - \varepsilon(Af, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)).\end{aligned}\tag{4.9}$$

Note that the limit of  $\mathcal{L}^\varepsilon \varphi^\varepsilon$  as  $\varepsilon$  goes to 0 does depend on  $n \in E$  with the term  $(fn, D\varphi_1(f))$ . Since the expected limit is  $\mathcal{L}\varphi$  where  $\varphi$  does not depend on  $n$ , we have to correct this term to cancel the dependence with respect to  $n$  of the limit. This is the aim of the second order correction  $\varphi_2$ . The right way to do so, given the mixing properties of the operator  $\mathcal{M}$ , is to subtract the mean value of this term under the invariant measure of the Markov process  $(g(t, f), m(t, n))_{t \geq 0}$  governed by  $\mathcal{M}$ . We write

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f, n) &= -\frac{1}{\varepsilon}(Af, D\varphi(f)) + \int_E (\bar{f}Fn, D\varphi_1(\bar{f}F)) d\nu(n) \\ &\quad + \mathcal{M}\varphi_2 + (fn, D\varphi_1(f)) - \int_E (\bar{f}Fn, D\varphi_1(\bar{f}F)) d\nu(n) \\ &\quad - (Af, D\varphi_1(f)) - \varepsilon(Af, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)),\end{aligned}$$

and we can now define  $\varphi_2$  as the solution of the well-posed Poisson equation

$$\mathcal{M}\varphi_2 = -(fn, D\varphi_1(f)) + \int_E (\bar{f}Fn, D\varphi_1(\bar{f}F)) d\nu(n).$$

Note that, thanks to the definition of  $\varphi_1$  given above, we can compute

$$(\bar{f}Fn, D\varphi_1(\bar{f}F)) = -(fnM^{-1}I(n), D\varphi(f)) - D^2\varphi(f)(fM^{-1}I(n), fn) =: q(f, n)$$

As a result, we easily have the following proposition.

**Proposition 4.2** (Second corrector). *Let  $\varphi \in C^3(L_{F^{-1}}^2)$  be a good test-function satisfying (4.1) and depending only on  $\bar{f}F$ . For any  $(f, n) \in L_{F^{-1}}^2 \times E$ , we define the second corrector  $\varphi_2$  as*

$$\varphi_2(f, n) := \mathbb{E} \int_0^\infty \left( \int_E (q(\bar{f}F, n) d\nu(n) - q(g(t, f), m(t, n))) dt, \right.$$

*which is well defined and satisfies the bounds*

$$(i) \quad |\varphi_2(f, n)| \lesssim C_\varphi(1 + \|f\|)^2, \quad (ii) \quad \|AD\varphi_2(f, n)\| \lesssim C_\varphi(1 + \|f\|).\tag{4.10}$$

The existence of  $\varphi_2$  is based on (2.9) and the bounds (4.10) are proved using (2.10) and (4.1).

### 4.1.4 Summary

The correctors  $\varphi_1$  and  $\varphi_2$  being defined as above in Propositions 4.1 and 4.2, we are finally led to

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f, n) &= -\frac{1}{\varepsilon}(Af, D\varphi(f)) + \int_E (\bar{f}Fn, D\varphi_1(\bar{f}F)) d\nu(n) \\ &\quad - (Af, D\varphi_1(f)) - \varepsilon(Af, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)).\end{aligned}$$

We are now able to define the limit generator  $\mathcal{L}$  as, for all  $\rho \in L^2(\mathbb{T}^N)$ ,

$$\begin{aligned} \mathcal{L}\varphi(\rho) := & (\operatorname{div}_x(\sigma(\rho)^{-1}K\nabla_x\rho)F, D\varphi(\rho F)) - \int_E (\rho F n M^{-1}I(n), D\varphi(\rho F)) \, d\nu(n) \\ & - \int_E D^2\varphi(\rho F)(\rho F M^{-1}I(n), \rho F n) \, d\nu(n), \end{aligned} \quad (4.11)$$

and we have shown the following equality

$$\begin{aligned} \mathcal{L}^\varepsilon \varphi^\varepsilon(f, n) = & \mathcal{L}\varphi(\bar{f}) - \frac{1}{\varepsilon}(Af, D\varphi(f)) - (\operatorname{div}_x(\sigma(\bar{f})^{-1}K\nabla_x\bar{f})F, D\varphi(\bar{f}F)) \\ & - (Af, D\varphi_1(f)) - \varepsilon(Af, D\varphi_2(f)) + \varepsilon(fn, D\varphi_2(f)). \end{aligned} \quad (4.12)$$

## 5 Uniform bound in $L^2_{F^{-1}}$

In this section, we prove a uniform estimate of the  $L^2_{F^{-1}}$  norm of the solution  $f^\varepsilon$  with respect to  $\varepsilon$ . To do so, we will again use the perturbed test functions method. The result is the following:

**Proposition 5.1.** *Let  $p \geq 1$  and  $f_0^\varepsilon \in D(A)$ . We have the two following bounds*

$$\mathbb{E} \sup_{t \in [0, T]} \|f_t^\varepsilon\|^p \lesssim 1, \quad (5.1)$$

$$\mathbb{E} \left( \int_0^T \|\sigma^{\frac{1}{2}}(\bar{f}_s^\varepsilon) L f_s^\varepsilon\|^2 \, ds \right)^p \lesssim \varepsilon^{2p}. \quad (5.2)$$

*Proof.* We set, for all  $f \in L^2_{F^{-1}}$ ,  $\varphi(f) := \frac{1}{2}\|f\|^2$ , which is easily seen to be a good test function. Then, with Proposition 3.1, the fact that  $A$  is skew-adjoint, (2.4), and the fact that  $\varphi$  does not depend on  $n \in E$ , we get for  $f \in D(A)$  and  $n \in E$ ,

$$\begin{aligned} \mathcal{L}^\varepsilon \varphi(f, n) = & -\frac{1}{\varepsilon}(Af, f) + \frac{1}{\varepsilon^2}(\sigma(\bar{f})Lf, f) + \frac{1}{\varepsilon}(fn, f) + \frac{1}{\varepsilon^2}M\varphi(f, n) \\ = & -\frac{1}{\varepsilon^2}\|\sigma^{\frac{1}{2}}(\bar{f})Lf\|^2 + \frac{1}{\varepsilon}(fn, f). \end{aligned}$$

The first term has a favourable behaviour for our purpose. The second term is more difficult to control and we correct  $\varphi$  thanks to the perturbed test-functions method to get rid of it: we recall the formal computations done in Section 4.1 and we set  $\varphi_1(f, n) = -(f, M^{-1}I(n)f)$  and  $\varphi^\varepsilon := \varphi(f, n) + \varepsilon\varphi_1$ . We can show that  $\varphi_1$  is a good test function with, thanks to Proposition 3.1,

$$\begin{aligned} \varepsilon \mathcal{L}^\varepsilon \varphi_1(f, n) = & -\frac{2}{\varepsilon}(\sigma(\bar{f})Lf, M^{-1}I(n)f) - 2(Af, M^{-1}I(n)f) \\ & - 2(fn, M^{-1}I(n)f) - \frac{1}{\varepsilon}(fn, f). \end{aligned}$$

As a consequence, we are led to

$$\begin{aligned} \mathcal{L}^\varepsilon \varphi^\varepsilon(f, n) = & -\frac{1}{\varepsilon^2}\|\sigma^{\frac{1}{2}}(\bar{f})Lf\|^2 - \frac{2}{\varepsilon}(\sigma(\bar{f})Lf, M^{-1}I(n)f) - 2(Af, M^{-1}I(n)f) \\ & - 2(fn, M^{-1}I(n)f). \end{aligned}$$

We use (2.10) and the hypothesis (H1) made on  $\sigma$  to bound the second term:

$$\begin{aligned} \frac{2}{\varepsilon}(\sigma(\bar{f})Lf, M^{-1}I(n)f) &\leq 2C_*(\sigma^*)^{\frac{1}{2}}\varepsilon^{-1}\|\sigma^{\frac{1}{2}}(\bar{f})Lf\|\|f\| \\ &\leq \frac{1}{2\varepsilon^2}\|\sigma^{\frac{1}{2}}(\bar{f})Lf\|^2 + 2C_*^2\sigma^*\|f\|^2. \end{aligned}$$

Furthermore, for the last two terms, we write

$$\begin{aligned} -2(Af, M^{-1}I(n)f) - 2(fn, M^{-1}I(n)f) &= (f^2, AM^{-1}I(n)) - 2(fn, M^{-1}I(n)f) \\ &\leq \|f\|^2\|a\|_{L^\infty(V)}C_* + 2C_*^2\|f\|^2. \end{aligned}$$

To sum up, we have proved that

$$\mathcal{L}^\varepsilon\varphi^\varepsilon(f, n) \lesssim -\frac{1}{2\varepsilon^2}\|\sigma^{\frac{1}{2}}(\bar{f})Lf\|^2 + \|f\|^2. \quad (5.3)$$

As in Proposition 3.1, since  $\varphi^\varepsilon$  is a good test function, we now define

$$M^\varepsilon(t) := \varphi^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) - \varphi^\varepsilon(f_0^\varepsilon, m_0^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(f_s^\varepsilon, m_s^\varepsilon) ds,$$

which is a continuous and integrable  $(\mathcal{F}_t^\varepsilon)_{t \geq 0}$  martingale. By definition of  $\varphi$ ,  $\varphi^\varepsilon$  and  $M^\varepsilon$ , we obtain

$$\frac{1}{2}\|f_t^\varepsilon\|^2 = \frac{1}{2}\|f_0^\varepsilon\|^2 - \varepsilon(\varphi_1(f_t^\varepsilon, m_t^\varepsilon) - \varphi_1(f_0^\varepsilon, m_0^\varepsilon)) + \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(f_s^\varepsilon, m_s^\varepsilon) ds + M^\varepsilon(t).$$

Since we have obviously  $|\varphi_1(f, n)| \lesssim \|f\|^2$ , we can write, with (5.3),

$$\|f_t^\varepsilon\|^2 \lesssim \|f_0^\varepsilon\|^2 + \varepsilon\|f_t^\varepsilon\| + \int_0^t -\frac{1}{2\varepsilon^2}\|\sigma^{\frac{1}{2}}(\bar{f}_s^\varepsilon)Lf_s^\varepsilon\|^2 + \|f_s^\varepsilon\|^2 ds + \sup_{t \in [0, T]} |M^\varepsilon(t)|,$$

i.e. for  $\varepsilon$  sufficiently small,

$$\int_0^t \frac{1}{2\varepsilon^2}\|\sigma^{\frac{1}{2}}(\bar{f}_s^\varepsilon)Lf_s^\varepsilon\|^2 ds + \|f_t^\varepsilon\|^2 \lesssim \|f_0^\varepsilon\|^2 + \int_0^t \|f_s^\varepsilon\|^2 ds + \sup_{t \in [0, T]} |M^\varepsilon(t)|,$$

and by Gronwall lemma,

$$\int_0^t \frac{1}{2\varepsilon^2}\|\sigma^{\frac{1}{2}}(\bar{f}_s^\varepsilon)Lf_s^\varepsilon\|^2 ds + \|f_t^\varepsilon\|^2 \lesssim \|f_0^\varepsilon\|^2 + \sup_{t \in [0, T]} |M^\varepsilon(t)|. \quad (5.4)$$

Note that  $|\varphi^\varepsilon|^2$  is a good test function with, thanks to (2.10) and (2.11),

$$|\mathcal{L}^\varepsilon|\varphi^\varepsilon|^2 - 2\varphi^\varepsilon\mathcal{L}^\varepsilon\varphi^\varepsilon| = |M|\varphi_1|^2 - 2\varphi_1M\varphi_1| \lesssim \|f\|^4,$$

and that, with Proposition 3.1, the quadratic variation of  $M^\varepsilon(t)$  is given by

$$\langle M^\varepsilon \rangle_t = \int_0^t (\mathcal{L}^\varepsilon|\varphi^\varepsilon|^2 - 2\varphi^\varepsilon\mathcal{L}^\varepsilon\varphi^\varepsilon)(f_s^\varepsilon, m_s^\varepsilon) ds.$$

As a result, with Burkholder-Davis-Gundy and Hölder inequalities, we get

$$\mathbb{E} \sup_{t \in [0, T]} |M^\varepsilon(t)|^p \lesssim \mathbb{E} |\langle M^\varepsilon \rangle_T|^{\frac{p}{2}} \lesssim \int_0^T \mathbb{E} \|f_s^\varepsilon\|^{2p} ds. \quad (5.5)$$

Neglecting the first (positive) term of the left-hand side in (5.4), we have

$$\mathbb{E} \|f_t^\varepsilon\|^{2p} \lesssim \mathbb{E} \|f_0^\varepsilon\|^{2p} + \mathbb{E} \sup_{t \in [0, T]} |M^\varepsilon(t)|^p,$$

so that we get

$$\mathbb{E} \|f_T^\varepsilon\|^{2p} \lesssim \mathbb{E} \|f_0^\varepsilon\|^{2p} + \int_0^T \mathbb{E} \|f_s^\varepsilon\|^{2p} ds,$$

and, by Gronwall lemma,

$$\mathbb{E} \|f_T^\varepsilon\|^{2p} \lesssim \mathbb{E} \|f_0^\varepsilon\|^{2p}. \quad (5.6)$$

This actually holds true for any  $t \in [0, T]$ . Thus, using (5.5) and (5.6) in (5.4) finally gives the expected bounds.  $\square$

**Remark** We define  $g^\varepsilon := f^\varepsilon - \rho^\varepsilon F = -L f^\varepsilon$ . Since we have  $\sigma \geq \sigma_*$ , the bound (5.2) gives that, for all  $p \geq 1$ ,

$$(\varepsilon^{-1} g^\varepsilon)_{\varepsilon > 0} \text{ is bounded in } L^p(\Omega; L^2(0, T; L_{F^{-1}}^2)). \quad (5.7)$$

In the sequel, we must deal with the non-linear term. To do so, we need some compactness in the space variable of the process  $(\rho^\varepsilon)_{\varepsilon > 0}$ . The following proposition is a first step to this purpose.

**Proposition 5.2.** *We assume that hypothesis (2.3) is satisfied. Let  $p \geq 1$  and  $s \in (0, \theta/2)$ . We have the bound*

$$\mathbb{E} \left( \int_0^T \|\rho_s^\varepsilon\|_{H^s(\mathbb{T}^N)}^2 ds \right)^p \lesssim 1. \quad (5.8)$$

*Proof.* Note that with  $\sigma \leq \sigma^*$ , the remark (5.7) and equation (1.1), we observe that

$$(\varepsilon \partial_t f^\varepsilon + a(v) \cdot \nabla_x f^\varepsilon - f^\varepsilon m^\varepsilon)_{\varepsilon > 0} \text{ is bounded in } L^p(\Omega; L^2(0, T; L_{F^{-1}}^2)).$$

Furthermore,  $(f^\varepsilon)_{\varepsilon > 0}$  is bounded in  $L^p(\Omega; L^2(0, T; L_{F^{-1}}^2))$  with (5.1) and  $|m^\varepsilon| \leq C_*$  so that

$$(\varepsilon \partial_t f^\varepsilon + a(v) \cdot \nabla_x f^\varepsilon)_{\varepsilon > 0} \text{ is bounded in } L^p(\Omega; L^2(0, T; L_{F^{-1}}^2)). \quad (5.9)$$

Then, thanks to (2.3), we apply an averaging lemma to conclude. Precisely, [10, Theorem 3.1] in the unstationary case applies a.s. with  $\beta = \gamma = 0$ ,  $p_1 = q_1 = p_2 = q_2 = 2$ ,  $a = 0$ ,  $k = \theta$  and

$$f = f^\varepsilon, \quad g = \varepsilon \partial_t f^\varepsilon + a(v) \cdot \nabla_x f^\varepsilon,$$

and gives the bound

$$\|\rho^\varepsilon\|_{B_{\infty, \infty}^{\frac{\theta}{2}, 2}} \leq C \|f^\varepsilon\|^{\frac{1}{2}} \|\varepsilon \partial_t f^\varepsilon + a(v) \cdot \nabla_x f^\varepsilon\|^{\frac{1}{2}} \quad \text{a.s.}$$

Since, for any  $s < \theta/2$ ,  $H^s \subset B_{\infty, \infty}^{\frac{\theta}{2}, 2}$ , it yields, for  $p \geq 1$ ,

$$\mathbb{E} \left( \int_0^T \|\rho_s^\varepsilon\|_{H^s}^2 ds \right)^p \leq C \mathbb{E} \left( \int_0^T \|f_s^\varepsilon\| \|\varepsilon \partial_t f_s^\varepsilon + a(v) \cdot \nabla_x f_s^\varepsilon\| ds \right)^p,$$

so that the result follows with Cauchy Schwarz inequality and (5.1) and (5.9). This concludes the proof.  $\square$

## 6 Tightness

We want to prove the convergence in law of the family  $(\rho^\varepsilon)_{\varepsilon>0}$ : in this section, we study the tightness of the processes  $(\rho^\varepsilon)_{\varepsilon>0}$  in the space  $C([0, T], H^{-\eta}(\mathbb{T}^N))$  where  $\eta > 0$ . In fact, this will not be sufficient to pass to the limit in the non-linear term. As a consequence, we also prove that  $(\rho^\varepsilon)_{\varepsilon>0}$  is tight in the space  $L^2(0, T; L^2(\mathbb{T}^N))$ .

**Proposition 6.1.** *Let  $\eta > 0$ . Then the sequence  $(\rho^\varepsilon)_{\varepsilon>0}$  is tight in the spaces  $C([0, T], H^{-\eta}(\mathbb{T}^N))$  and  $L^2(0, T; L^2(\mathbb{T}^N))$ .*

*Proof. Step 1: control of the modulus of continuity of  $\rho^\varepsilon$  in  $H^{-\eta}(\mathbb{T}^N)$ .* Let  $\eta > 0$  be fixed. For any  $\delta > 0$ , we define

$$w(\rho, \delta) := \sup_{|t-s|<\delta} \|\rho(t) - \rho(s)\|_{H^{-\eta}(\mathbb{T}^N)}$$

the modulus of continuity of a function  $\rho \in C([0, T], H^{-\eta}(\mathbb{T}^N))$ . In this first step of the proof, we want to obtain the following bound

$$\mathbb{E}w(\rho^\varepsilon, \delta) \lesssim \varepsilon + \delta^\tau, \quad (6.1)$$

for some positive  $\tau$ . To do so, we use the perturbed test-functions method. Let  $(p_j)_{j \in \mathbb{N}^N}$  the Fourier orthonormal basis of  $L^2(\mathbb{T}^N)$  and  $J$  the operator

$$J := (\mathbf{I} - \Delta_x)^{-\frac{1}{2}}.$$

Let  $j \in \mathbb{N}^N$ . We set

$$\varphi_j(f) := (f, p_j F), \quad f \in L_{F^{-1}}^2,$$

and we define the first order corrections by, see Section 4.1,

$$\varphi_{1,j}(f, n) := -(f M^{-1} I(n), p_j F), \quad (f, n) \in L_{F^{-1}}^2 \times E.$$

We finally define  $\varphi_j^\varepsilon := \varphi_j + \varepsilon \varphi_{1,j}$ , which is easily seen to be a good test-function, so that, thanks to Proposition 3.1, we consider the continuous martingales

$$M_j^\varepsilon(t) := \varphi_j^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) - \varphi_j^\varepsilon(f_0^\varepsilon, m_0^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon \varphi_j^\varepsilon(f_s^\varepsilon, m_s^\varepsilon) ds.$$

We also define,

$$\theta_j^\varepsilon(t) := \varphi_j(f_0^\varepsilon) + \int_0^t \mathcal{L}^\varepsilon \varphi_j^\varepsilon(f_s^\varepsilon, m_s^\varepsilon) ds + M_j^\varepsilon(t).$$

Note that

$$\theta_j^\varepsilon(t) = \varphi_j(f_t^\varepsilon) + \varepsilon(\varphi_{1,j}(f_t^\varepsilon, m_t^\varepsilon) - \varphi_{1,j}(f_0^\varepsilon, m_0^\varepsilon)), \quad (6.2)$$

so that, with the definitions of  $\varphi_j$  and  $\varphi_{1,j}$ , Cauchy-Schwarz inequality, we easily get

$$|\theta_j^\varepsilon(t)| \lesssim \sup_{t \in [0, T]} \|f^\varepsilon(t)\| \|p_j\|_{L_x^2} = \sup_{t \in [0, T]} \|f^\varepsilon(t)\|.$$

Hence, by the uniform  $L_{F^{-1}}^2$  bound (5.1),

$$\mathbb{E} \sup_{t \in [0, T]} |\theta_j^\varepsilon(t)| \lesssim 1. \quad (6.3)$$

With (6.2) and the uniform  $L_{F-1}^2$  bound (5.1), we also deduce

$$\mathbb{E} \sup_{t \in [0, T]} |\varphi_j(\rho_t^\varepsilon) - \theta_j^\varepsilon(t)| \lesssim \varepsilon. \quad (6.4)$$

From now on, we fix  $\gamma > N/2 + 2$  and we remark that, by (6.3), a.s. and for all  $t \in [0, T]$ , the series defined by  $u_t^\varepsilon := \sum_{j \in \mathbb{N}^N} \theta_j^\varepsilon(t) J^\gamma p_j$  converges in  $L^2(\mathbb{T}^N)$ . We then set

$$\theta^\varepsilon(t) := J^{-\gamma} \sum_{j \in \mathbb{N}^N} \theta_j^\varepsilon(t) J^\gamma p_j,$$

which exists a.s. and for all  $t \in [0, T]$  in  $H^{-\gamma}(\mathbb{T}^N)$ . And with (6.4), we obtain

$$\mathbb{E} \sup_{t \in [0, T]} \|\rho^\varepsilon(t) - \theta^\varepsilon(t)\|_{H^{-\gamma}(\mathbb{T}^N)} \lesssim \varepsilon. \quad (6.5)$$

Actually, by interpolation, the continuous embedding  $L^2(\mathbb{T}^N) \subset H^{-\eta}(\mathbb{T}^N)$  and the uniform  $L_{F-1}^2$  bound (5.1), we have

$$\mathbb{E} \sup_{|t-s| < \delta} \|\rho(t) - \rho(s)\|_{H^{-\eta^b}} \leq \mathbb{E} \sup_{|t-s| < \delta} \|\rho(t) - \rho(s)\|_{H^{-\eta^\sharp}}^v$$

for a certain  $v > 0$  if  $\eta^\sharp > \eta^b > 0$ . As a result, it is indeed sufficient to work with  $\eta = \gamma$ . In view of (6.5), we first want to obtain an estimate of the increments of  $\theta^\varepsilon$ . We have, for  $j \in \mathbb{N}^N$  and  $0 \leq s \leq t \leq T$ ,

$$\theta_j^\varepsilon(t) - \theta_j^\varepsilon(s) = \int_s^t \mathcal{L}^\varepsilon \varphi_j^\varepsilon(f_\sigma^\varepsilon, m_\sigma^\varepsilon) d\sigma + M_j^\varepsilon(t) - M_j^\varepsilon(s). \quad (6.6)$$

We then control the two terms on the right-hand side of (6.6). Let us begin with the first one. Note that, since  $D\varphi_j(f) \equiv p_j F$  and  $D\varphi_{1,j}(f) \equiv -M^{-1}I(n)p_j F$ , we obtain thanks to (4.9) with  $\varphi_2 \equiv 0$ ,

$$\mathcal{L}^\varepsilon \varphi_j^\varepsilon(f_\sigma^\varepsilon, m_\sigma^\varepsilon) = -\frac{1}{\varepsilon} (A f_\sigma^\varepsilon, p_j F) + (A f_\sigma^\varepsilon, M^{-1}I(m_\sigma^\varepsilon)p_j F) - (f_\sigma^\varepsilon m_\sigma^\varepsilon, M^{-1}I(m_\sigma^\varepsilon)p_j F).$$

Since, with (2.2), we have  $\overline{a(v)f_\sigma^\varepsilon} = \overline{a(v)g_\sigma^\varepsilon}$  where  $g^\varepsilon$  has been defined previously as  $g^\varepsilon := f^\varepsilon - \rho^\varepsilon F$ , we can write

$$(A f_\sigma^\varepsilon, p_j F) = \int_{\mathbb{T}^N} \operatorname{div}_x(\overline{a(v)f_\sigma^\varepsilon}) p_j dx = \int_{\mathbb{T}^N} \operatorname{div}_x(\overline{a(v)g_\sigma^\varepsilon}) p_j dx = (A g_\sigma^\varepsilon, p_j F)$$

and, as a consequence, since  $a$  is bounded, we are led to

$$\frac{1}{\varepsilon} (A f_\sigma^\varepsilon, p_j F) \lesssim \|\varepsilon^{-1} g_\sigma^\varepsilon\| \|\nabla_x p_j\|_{L^2}.$$

Similarly, we can show that

$$(A f_\sigma^\varepsilon, M^{-1}I(m_\sigma^\varepsilon)p_j F) \lesssim \|g_\sigma^\varepsilon\| (1 + \|\nabla_x p_j\|_{L^2}).$$

Since we have obviously  $(f_\sigma^\varepsilon m_\sigma^\varepsilon, M^{-1}I(m_\sigma^\varepsilon)p_j F) \lesssim \|f_\sigma^\varepsilon\|$ , we can conclude that

$$|\mathcal{L}^\varepsilon \varphi_j^\varepsilon(f_\sigma^\varepsilon, m_\sigma^\varepsilon)| \lesssim C_j [\|\varepsilon^{-1} g_\sigma^\varepsilon\| + \|g_\sigma^\varepsilon\| + \|f_\sigma^\varepsilon\|], \quad (6.7)$$



where  $C_j := 1 + \|\nabla_x p_j\|_{L^2} \leq 1 + |j|$ . Thanks to (5.1) and (5.7) with  $p = 4$ , we have that  $(\varepsilon^{-1}g^\varepsilon)_{\varepsilon>0}$ ,  $(g^\varepsilon)_{\varepsilon>0}$  and  $(f^\varepsilon)_{\varepsilon>0}$  are bounded in  $L^4(\Omega; L^2(0, T; L_{F^{-1}}^2))$ . As a consequence, (6.7) and an application of Hölder's inequality gives

$$\mathbb{E} \left| \int_s^t \mathcal{L}^\varepsilon \varphi_j^\varepsilon(f_\sigma^\varepsilon, m_\sigma^\varepsilon) d\sigma \right|^4 \lesssim C_j^4 |t - s|^2.$$

Furthermore, using Burkholder-Davis-Gundy inequality, we can control the second term of the right-hand side of (6.6) as

$$\mathbb{E}|M_j^\varepsilon(t) - M_j^\varepsilon(s)|^4 \lesssim \mathbb{E}|\langle M_j^\varepsilon \rangle_t - \langle M_j^\varepsilon \rangle_s|^2,$$

where the quadratic variation  $\langle M_j^\varepsilon \rangle$  is given by

$$\langle M_j^\varepsilon \rangle_t = \int_0^t (M|\varphi_{1,j}|^2 - 2\varphi_{1,j}M\varphi_{1,j})(f_s^\varepsilon, m_s^\varepsilon) ds.$$

With the definition of  $\varphi_{1,j}$ , (2.10), (2.11) and the uniform  $L_{F^{-1}}^2$  bound (5.1), it is now easy to get

$$\mathbb{E}|M_j^\varepsilon(t) - M_j^\varepsilon(s)|^4 \lesssim |t - s|^2.$$

Finally we have  $\mathbb{E}|\theta_j^\varepsilon(t) - \theta_j^\varepsilon(s)|^4 \lesssim (1 + |j|^4)|t - s|^2$ . Since we took  $\gamma > N/2 + 2$ , we can conclude that

$$\mathbb{E}\|\theta^\varepsilon(t) - \theta^\varepsilon(s)\|_{H^{-\gamma}(\mathbb{T}^N)}^4 \lesssim |t - s|^2.$$

It easily follows that, for  $v < 1/2$ ,

$$\mathbb{E}\|\theta^\varepsilon\|_{W^{v,4}(0,T,H^{-\gamma}(\mathbb{T}^N))}^4 \lesssim 1$$

and by the embedding

$$W^{v,4}(0,T,H^{-\gamma}(\mathbb{T}^N)) \subset C^\tau(0,T,H^{-\gamma}(\mathbb{T}^N)), \quad \tau < v - \frac{1}{4},$$

we obtain that  $\mathbb{E}w(\theta^\varepsilon, \delta) \lesssim \delta^\tau$  for a certain positive  $\tau$ . Finally, with (6.5), we can now conclude the first step of the proof since

$$\mathbb{E}w(\rho^\varepsilon, \delta) \leq 2\mathbb{E} \sup_{t \in [0,T]} \|\rho_t^\varepsilon - \theta_t^\varepsilon\|_{H^{-\gamma}(\mathbb{T}^N)} + \mathbb{E}w(\theta^\varepsilon, \delta) \lesssim \varepsilon + \delta^\tau. \quad (6.8)$$

*Step 2: tightness in  $C([0, T]; H^{-\eta}(\mathbb{T}^N))$ .* Since the embedding  $L^2(\mathbb{T}^N) \subset H^{-\eta}(\mathbb{T}^N)$  is compact, and by Ascoli's Theorem, the set

$$K_R := \left\{ \rho \in C([0, T], H^{-\eta}(\mathbb{T}^N)), \sup_{t \in [0, T]} \|\rho\|_{L^2(\mathbb{T}^N)} \leq R, w(\rho, \delta) < \varepsilon(\delta) \right\},$$

where  $R > 0$  and  $\varepsilon(\delta) \rightarrow 0$  when  $\delta \rightarrow 0$ , is compact in  $C([0, T], H^{-\eta}(\mathbb{T}^N))$ . By Prokhorov's Theorem, the tightness of  $(\rho^\varepsilon)_{\varepsilon>0}$  in  $C([0, T], H^{-\eta}(\mathbb{T}^N))$  will follow if we prove that for all  $\sigma > 0$ , there exists  $R > 0$  such that

$$\mathbb{P}(\sup_{t \in [0, T]} \|\rho^\varepsilon\|_{L^2(\mathbb{T}^N)} > R) < \sigma, \quad (6.9)$$

and

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(w(\rho^\varepsilon, \delta) > \sigma) = 0. \quad (6.10)$$

With Markov's inequality and the uniform  $L^2_{F^{-1}}$  bound (5.1), we have

$$\mathbb{P}(\sup_{t \in [0, T]} \|\rho^\varepsilon\|_{L^2(\mathbb{T}^N)} > R) \leq \mathbb{P}(\sup_{t \in [0, T]} \|f^\varepsilon\| > R) \lesssim R^{-1},$$

which gives (6.9). And we deduce (6.10) by Markov's inequality and the bound (6.1) since

$$\begin{aligned} \lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(w(\rho^\varepsilon, \delta) > \sigma) &\leq \lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \sigma^{-1} \mathbb{E}w(\rho^\varepsilon, \delta) \\ &\lesssim \lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \sigma^{-1} (\varepsilon + \delta^\tau) = 0. \end{aligned}$$

*Step 3: tightness in  $L^2(0, T; L^2(\mathbb{T}^N))$ .* Similarly, due to [12, Theorem 5], the set

$$K_R := \left\{ \rho \in L^2(0, T; L^2(\mathbb{T}^N)), \int_0^T \|\rho_t\|_{H^s(\mathbb{T}^N)}^2 dt \leq R, w(\rho, \delta) < \varepsilon(\delta) \right\},$$

where  $R > 0$ ,  $s > 0$  and  $\varepsilon(\delta) \rightarrow 0$  when  $\delta \rightarrow 0$ , is compact in  $L^2(0, T; L^2(\mathbb{T}^N))$ . By Prokhorov's Theorem, the tightness of  $(\rho^\varepsilon)_{\varepsilon > 0}$  in  $L^2(0, T; L^2(\mathbb{T}^N))$  will follow if we prove that for all  $\sigma > 0$ , there exists  $R > 0$  such that

$$\mathbb{P}(\int_0^T \|\rho_t\|_{H^s(\mathbb{T}^N)}^2 dt > R) < \sigma, \quad (6.11)$$

and

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(w(\rho^\varepsilon, \delta) > \sigma) = 0. \quad (6.12)$$

But (6.11) and (6.12) are consequences of Markov's inequality and the bounds (5.8) with  $p = 1$  and (6.1) so that the proof is complete.  $\square$

## 7 Convergence

We conclude here the proof of Theorem 2.2. The idea is now, by the tightness result and Prokhorov Theorem, to take a subsequence of  $(\rho^\varepsilon)_{\varepsilon > 0}$  that converges in law to some probability measure. Then we show that this limiting probability is actually uniquely determined by the limit generator  $\mathcal{L}$  defined above.

We fix  $\eta > 0$ . By Proposition 6.1 and Prokhorov's Theorem, there is a subsequence of  $(\rho^\varepsilon)_{\varepsilon > 0}$ , still denoted  $(\rho^\varepsilon)_{\varepsilon > 0}$ , and a probability measure  $P$  on the spaces  $C([0, T], H^{-\eta})$  and  $L^2(0, T; L^2)$  such that

$$P^\varepsilon \rightarrow P \text{ weakly in } C([0, T], H^{-\eta}) \text{ and } L^2(0, T; L^2),$$

where  $P^\varepsilon$  stands for the law of  $\rho^\varepsilon$ . We now identify the probability measure  $P$ .

Since the spaces  $C([0, T], H^{-\eta})$  and  $L^2(0, T; L^2)$  are separable, we can apply Skorohod representation Theorem [3], so that there exists a new probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  and random variables

$$\tilde{\rho}^\varepsilon, \tilde{\rho} : \tilde{\Omega} \rightarrow C([0, T], H^{-\eta}) \cap L^2(0, T; L^2),$$

with respective law  $P^\varepsilon$  and  $P$  such that  $\tilde{\rho}^\varepsilon \rightarrow \tilde{\rho}$  in  $C([0, T], H^{-\eta})$  and  $L^2(0, T; L^2)$   $\tilde{\mathbb{P}}$ -a.s. In the sequel, for the sake of clarity, we do not write any more the tildes.

Note that, with (5.7), we can also suppose that  $\varepsilon^{-1}g^\varepsilon$  converges to some  $g$  weakly in the space  $L^2(\Omega; L^2(0, T; L^2_{F^{-1}}))$ . Similarly, with (2.10), we assume that  $m^\varepsilon$  converges to  $m$  weakly in  $L^2(\Omega; L^2(0, T; L^2_{F^{-1}}))$ . Before going on the proof, we want to identify the weak limit  $g$  of  $\varepsilon^{-1}g^\varepsilon$ .

**Lemma 7.1.** *In  $L^2(\Omega; L^2(0, T; L^2))$ , we have the relation*

$$\overline{a(v)g} = -\sigma(\rho)^{-1}K\nabla_x\rho.$$

*Proof.* We define  $D_T := (0, T) \times \mathbb{T}^N$ . Since  $f^\varepsilon$  satisfies equation (1.1), we can write, for any  $\psi \in C_c^\infty(D_T)$  and  $\theta \in L^\infty(V \times \Omega; \mathbb{R}^N)$ ,

$$\begin{aligned} \mathbb{E} \int_{D_T \times V} f^\varepsilon F^{-1} (-\varepsilon \partial_t \psi - a \cdot \nabla_x \psi) \theta &= \mathbb{E} \int_{D_T \times V} \frac{1}{\varepsilon} \sigma(\overline{f^\varepsilon}) L f^\varepsilon F^{-1} \psi \theta \\ &\quad + \mathbb{E} \int_{D_T \times V} m^\varepsilon f^\varepsilon F^{-1} \psi \theta. \end{aligned}$$

We recall that we set  $g^\varepsilon := f^\varepsilon - \rho^\varepsilon F$  and that  $L f^\varepsilon = L g^\varepsilon$  so that we have

$$\begin{aligned} \mathbb{E} \int_{D_T \times V} -\varepsilon f^\varepsilon F^{-1} \partial_t \psi \theta - \rho^\varepsilon a \cdot \nabla_x \psi \theta - g^\varepsilon F^{-1} a \cdot \nabla_x \psi \theta \\ = \mathbb{E} \int_{D_T \times V} \sigma(\rho^\varepsilon) L(\varepsilon^{-1} g^\varepsilon) F^{-1} \psi \theta + \mathbb{E} \int_{D_T \times V} m^\varepsilon f^\varepsilon F^{-1} \psi \theta. \end{aligned}$$

Since  $(f^\varepsilon)_{\varepsilon>0}$  and  $(\varepsilon^{-1}g^\varepsilon)_{\varepsilon>0}$  are bounded in  $L^2(\Omega; L^2(0, T; L^2_{F^{-1}}))$  by (5.1) and (5.7), and with the  $\mathbb{P}$ -a.s. convergence  $\rho^\varepsilon \rightarrow \rho$  in  $L^2(0, T; L^2_{F^{-1}})$  coupled with the uniform integrability of the family  $(\rho^\varepsilon)_{\varepsilon>0}$  obtained with (5.1), we have that the left-hand side of the previous equality actually converges as  $\varepsilon \rightarrow 0$  to

$$\mathbb{E} \int_{D_T \times V} -\rho a \cdot \nabla_x \psi \theta.$$

Note that,  $\mathbb{P}$ -a.s., we have the following convergences in  $L^2(0, T; L^2_{F^{-1}})$

$$\sigma(\rho^\varepsilon) \rightarrow \sigma(\rho), \quad L(\varepsilon^{-1}g^\varepsilon) \rightharpoonup Lg, \quad f^\varepsilon \rightarrow \rho F, \quad m^\varepsilon \rightharpoonup m,$$

where the first convergence is justified by the Lipschitz continuity of  $\sigma$ . As a result, since all the quantities above are uniformly integrable with respect to  $\varepsilon$  thanks to (5.1), (5.7) and (2.10), the right-hand side of the previous equality converges as  $\varepsilon \rightarrow 0$  to

$$\mathbb{E} \int_{D_T \times V} \sigma(\rho) L(g) F^{-1} \psi \theta + \mathbb{E} \int_{D_T \times V} m \rho \psi \theta.$$

Thus, we have

$$\mathbb{E} \int_{D_T \times V} -\rho a \cdot \nabla_x \psi \theta = \mathbb{E} \int_{D_T \times V} \sigma(\rho) L(g) F^{-1} \psi \theta + \mathbb{E} \int_{D_T \times V} m \rho \psi \theta.$$

Let  $\xi$  be an arbitrary bounded measurable function on  $\Omega$ . We now set  $\theta(v, \omega) = a(v)F(v)\xi(\omega)$ ; note that we do have  $\theta \in L^\infty(V \times \Omega, \mathbb{R}^N)$ . With (2.2) and the relation  $Lg = \bar{g}F - g$ , we obtain

$$-\mathbb{E} \int_{D_T \times V} \rho a \cdot \nabla_x \psi a F = -\mathbb{E} \int_{D_T \times V} \sigma(\rho) g a(v) \psi.$$

Since this relation holds for any  $\xi \in L^\infty(\Omega)$  and  $\psi \in C_c^\infty(D_T)$ , we deduce that  $\nabla_x \rho \in L^2(\Omega, L^2(D_T))$  and that

$$\overline{a(v)g} = -\sigma(\rho)^{-1} K \nabla_x \rho,$$

and this concludes the proof.  $\square$

Let  $\varphi \in C^3(L_{F-1}^2)$  a good test-function satisfying (4.1). We define  $\varphi^\varepsilon$  as in Section 4.1. Since  $\varphi^\varepsilon$  is a good test-function, we have that

$$\varphi^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) - \varphi^\varepsilon(f_0^\varepsilon, m_0^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon \varphi^\varepsilon(f_s^\varepsilon, m_s^\varepsilon) ds, \quad t \in [0, T],$$

is a continuous martingale for the filtration generated by  $(f_s^\varepsilon)_{s \in [0, T]}$ . As a result, if  $\Psi$  denotes a continuous and bounded function from  $L^2(\mathbb{T}^N)^n$  to  $\mathbb{R}$ , we have

$$\mathbb{E} \left[ \left( \varphi^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) - \varphi^\varepsilon(f_s^\varepsilon, m_s^\varepsilon) - \int_s^t \mathcal{L}^\varepsilon \varphi^\varepsilon(f_u^\varepsilon, m_u^\varepsilon) du \right) \Psi(\rho_{s_1}^\varepsilon, \dots, \rho_{s_n}^\varepsilon) \right] = 0, \quad (7.1)$$

for any  $0 \leq s_1 \leq \dots \leq s_n \leq s \leq t$ . Our final purpose is to pass to the limit  $\varepsilon \rightarrow 0$  in (7.1). In the sequel, we assume that the function  $\varphi$  and  $\Psi$  are also continuous on the space  $H^{-\eta}$ , which is always possible with an approximation argument: it suffices to consider  $\varphi_r := \varphi((I - r\Delta_x)^{-\frac{\eta}{2}} \cdot)$  and  $\Psi_r := \Psi((I - r\Delta_x)^{-\frac{\eta}{2}} \cdot, \dots, (I - r\Delta_x)^{-\frac{\eta}{2}} \cdot)$  as  $r \rightarrow 0$ . With (4.12), we divide the left-hand side of (7.1) in four parts. Precisely, we define, for  $i \in \{1, \dots, 4\}$

$$\begin{aligned} \tau_1^\varepsilon &:= \varphi^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) - \varphi^\varepsilon(f_s^\varepsilon, m_s^\varepsilon), \\ \tau_2^\varepsilon &:= \int_s^t \mathcal{L} \varphi(\rho_u^\varepsilon) du, \\ \tau_3^\varepsilon &:= \int_s^t -\frac{1}{\varepsilon} (Af_u^\varepsilon, D\varphi(f_u^\varepsilon)) - (\operatorname{div}_x(\sigma(\rho_u^\varepsilon)^{-1} K \nabla_x \rho_u^\varepsilon) F, D\varphi(\rho_u^\varepsilon F)) du, \\ \tau_4^\varepsilon &:= \int_s^t -(Af_u^\varepsilon, D\varphi_1(f_u^\varepsilon)) - \varepsilon (Af_u^\varepsilon, D\varphi_2(f_u^\varepsilon)) + \varepsilon (f_u^\varepsilon m_u^\varepsilon, D\varphi_2(f_u^\varepsilon)) du. \end{aligned}$$

*Study of  $\tau_1^\varepsilon$ .* We recall that  $\varphi^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) = \varphi(\rho_t^\varepsilon F) + \varepsilon \varphi_1(f_t^\varepsilon, m_t^\varepsilon) + \varepsilon^2 \varphi_2(f_t^\varepsilon, m_t^\varepsilon)$  so that, with the  $\mathbb{P}$ -a.s. convergence of  $\rho^\varepsilon$  to  $\rho$  in  $C([0, T], H^{-\eta})$  and the bounds (i) of (4.8) and (4.10), we have that  $\tau_1^\varepsilon$  converges  $\mathbb{P}$ -a.s. to  $\varphi(\rho_t F) - \varphi(\rho_s F)$  as  $\varepsilon$  goes to 0. Furthermore, with the continuity of  $\Psi$  in  $H^{-\eta}$ , we also have that  $\Psi(\rho_{s_1}^\varepsilon, \dots, \rho_{s_n}^\varepsilon)$  converges  $\mathbb{P}$ -a.s. to  $\Psi(\rho_{s_1}, \dots, \rho_{s_n})$ . Finally, since the family  $\tau_1^\varepsilon \Psi(\rho_{s_1}^\varepsilon, \dots, \rho_{s_n}^\varepsilon)$  is uniformly integrable with respect to  $\varepsilon$  thanks to (4.1), the bounds (i) of (4.8) and (4.10) and the uniform  $L_{F-1}^2$  bound (5.1), we have that

$$\mathbb{E}[\tau_1^\varepsilon \Psi(\rho_{s_1}^\varepsilon, \dots, \rho_{s_n}^\varepsilon)] \rightarrow \mathbb{E}[(\varphi(\rho_t F) - \varphi(\rho_s F)) \Psi(\rho_{s_1}, \dots, \rho_{s_n})].$$

*Study of  $\tau_2^\varepsilon$ .* We recall, with (4.11), that

$$\begin{aligned}\mathcal{L}\varphi(\rho_u^\varepsilon) &= (\operatorname{div}_x(\sigma(\rho_u^\varepsilon)^{-1}K\nabla_x\rho_u^\varepsilon)F, D\varphi(\rho_u^\varepsilon F)) - \int_E (\rho_u^\varepsilon F n M^{-1}I(n), D\varphi(\rho_u^\varepsilon F)) \, d\nu(n) \\ &\quad - \int_E D^2\varphi(\rho_u^\varepsilon F)(\rho_u^\varepsilon F M^{-1}I(n), \rho_u^\varepsilon F n) \, d\nu(n).\end{aligned}$$

Thanks to the  $\mathbb{P}$ -a.s. convergence of  $\rho^\varepsilon$  to  $\rho$  in  $L^2(0, T; L^2)$  and with  $\varphi \in C^3(L_{F^{-1}}^2)$ , we can pass to the limit  $\varepsilon \rightarrow 0$  in the term

$$\int_s^t \int_E -(\rho_u^\varepsilon F n M^{-1}I(n), D\varphi(\rho_u^\varepsilon F)) - D^2\varphi(\rho_u^\varepsilon F)(\rho_u^\varepsilon F M^{-1}I(n), \rho_u^\varepsilon F n) \, d\nu(n) \, du.$$

Regarding the first term of  $\mathcal{L}\varphi(\rho_u^\varepsilon)$ , we introduce

$$G(\rho) := \int_0^\rho \frac{dy}{\sigma(y)},$$

which is, thanks to the hypothesis (H1) made on  $\sigma$ , Lipschitz continuous on  $L^2(\mathbb{T}^N)$ . Now the first term of  $\mathcal{L}\varphi(\rho_u^\varepsilon)$  writes

$$(\operatorname{div}_x(\sigma(\rho_u^\varepsilon)^{-1}K\nabla_x\rho_u^\varepsilon)F, D\varphi(\rho_u^\varepsilon F)) = (\operatorname{div}_x\nabla_x G(\rho_u^\varepsilon)F, D\varphi(\rho_u^\varepsilon F)).$$

Furthermore, with (4.1), the mapping  $\rho \mapsto \partial_{x_i, x_j}^2 D\varphi(\rho F)$  is continuous on  $L^2(\mathbb{T}^N)$ . As a result, we can now pass to the limit in the term

$$\int_s^t (\operatorname{div}_x(\sigma(\rho_u^\varepsilon)^{-1}K\nabla_x\rho_u^\varepsilon)F, D\varphi(\rho_u^\varepsilon F)) \, du.$$

To sum up, we obtain that  $\tau_2^\varepsilon$  converges  $\mathbb{P}$ -a.s. to  $\int_s^t \mathcal{L}\varphi(\rho_u) \, du$  as  $\varepsilon$  goes to 0. Finally, since the family  $\tau_2^\varepsilon \Psi(\rho_{s_1}^\varepsilon, \dots, \rho_{s_n}^\varepsilon)$  is uniformly integrable with respect to  $\varepsilon$  thanks to (4.1) and the uniform  $L_{F^{-1}}^2$  bound (5.1), we have that

$$\mathbb{E}[\tau_2^\varepsilon \Psi(\rho_{s_1}^\varepsilon, \dots, \rho_{s_n}^\varepsilon)] \rightarrow \mathbb{E}\left[\left(\int_s^t \mathcal{L}\varphi(\rho_u) \, du\right) \Psi(\rho_{s_1}, \dots, \rho_{s_n})\right].$$

*Study of  $\tau_3^\varepsilon$ .* First of all, we observe that, with the decomposition  $f^\varepsilon = \rho^\varepsilon F + g^\varepsilon$ , (4.7) and (2.2),

$$-\varepsilon^{-1}(Af_u^\varepsilon, D\varphi(f_u^\varepsilon)) = -\varepsilon^{-1}(Ag_u^\varepsilon, D\varphi(f_u^\varepsilon)),$$

so that, with the  $\mathbb{P}$ -a.s. convergences in  $L^2(0, T; L^2)$

$$\varepsilon^{-1}g^\varepsilon \rightharpoonup g, \quad \rho^\varepsilon \rightarrow \rho,$$

and the continuity of the mapping  $\rho \mapsto AD\varphi(\rho F)$  thanks to (4.1), we obtain that the first term of  $\tau_3^\varepsilon$  converges  $\mathbb{P}$ -a.s. to

$$-\int_s^t (\overline{Ag_u}F, D\varphi(\rho_u F)) \, du.$$

And, with Lemma 7.1, this term writes

$$\int_s^t (\operatorname{div}_x(\sigma(\rho_u)^{-1} K \nabla_x \rho_u) F, D\varphi(\rho_u F)) \, du. \quad (7.2)$$

Furthermore, similarly as the case of  $\tau_2^\varepsilon$ , we have that the second term of  $\tau_3^\varepsilon$  converges  $\mathbb{P}$ -a.s. to the opposite of (7.2). As a result,  $\tau_3^\varepsilon$  converges  $\mathbb{P}$ -a.s. to 0. Finally, since the family  $\tau_3^\varepsilon \Psi(\rho_{s_1}^\varepsilon, \dots, \rho_{s_n}^\varepsilon)$  is uniformly integrable with respect to  $\varepsilon$  thanks to (4.1), the uniform  $L_{F^{-1}}^2$  bound (5.1) and the bound (5.7) on  $(\varepsilon^{-1} g^\varepsilon)_{\varepsilon>0}$ , we have that

$$\mathbb{E}[\tau_3^\varepsilon \Psi(\rho_{s_1}^\varepsilon, \dots, \rho_{s_n}^\varepsilon)] \rightarrow 0.$$

*Study of  $\tau_4^\varepsilon$ .* If we transform the two first terms of  $\tau_4^\varepsilon$  exactly as we do for the first term of  $\tau_3^\varepsilon$ , it is then easy, using the uniform bounds (5.1) and (5.7) and the bounds (ii) of (4.8) and (4.10), to get

$$\mathbb{E}[\tau_4^\varepsilon \Psi(\rho_{s_1}^\varepsilon, \dots, \rho_{s_n}^\varepsilon)] = O(\varepsilon).$$

To sum up, we can pass to the limit  $\varepsilon \rightarrow 0$  in (7.1) to obtain

$$\mathbb{E} \left[ \left( \varphi(\rho_t F) - \varphi(\rho_s F) - \int_s^t \mathcal{L} \varphi(\rho_u) \, du \right) \Psi(\rho_{s_1}, \dots, \rho_{s_n}) \right] = 0. \quad (7.3)$$

We recall that this is valid for all  $n \in \mathbb{N}$ ,  $0 \leq s_1 \leq \dots \leq s_n \leq s \leq t \in [0, T]$  and all  $\Psi$  continuous and bounded function on  $L^2(\mathbb{T}^N)^n$ . Now, let  $\xi$  be a smooth function on  $L^2(\mathbb{T}^N)$ . We choose  $\varphi(f) = (f, \xi F)$ . We can easily verify that  $\varphi$  and  $|\varphi|^2$  belong to  $C^3(L_{F^{-1}}^2)$  and that they are good test-function satisfying (4.1). Thus, we obtain that

$$\begin{aligned} N_t &:= \varphi(\rho_t F) - \varphi(\rho_0 F) - \int_0^t \mathcal{L} \varphi(\rho_u) \, du, \quad t \in [0, T], \\ |\varphi|^2(\rho_t F) - |\varphi|^2(\rho_0 F) - \int_0^t \mathcal{L} |\varphi|^2(\rho_u) \, du, \quad t \in [0, T], \end{aligned}$$

are continuous martingales with respect to the filtration generated by  $(\rho_s)_{s \in [0, T]}$ . It implies (see appendix 6.9 in [9]) that the quadratic variation of  $N$  is given by

$$\langle N \rangle_t = \int_0^t [\mathcal{L} |\varphi|^2(\rho_u) - 2\varphi(\rho_u) \mathcal{L} \varphi(\rho_u)] \, du, \quad t \in [0, T].$$

Furthermore, we have

$$\begin{aligned} \mathcal{L} |\varphi|^2(\rho_u) - 2\varphi(\rho_u) \mathcal{L} \varphi(\rho_u) &= -2 \int_E (\rho_u F n, \xi F) (\rho_u F M^{-1} I(n), \xi F) \, d\nu(n) \\ &= 2 \mathbb{E} \int_0^\infty (\rho_u F m_0, \xi F) (\rho_u F m_t, \xi F) \, dt \\ &= \mathbb{E} \int_{\mathbb{R}} (\rho_u F m_0, \xi F) (\rho_u F m_t, \xi F) \, dt \\ &= \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} \rho_u(x) \xi(x) \rho_u(y) \xi(y) k(x, y) \, dx dy \\ &= \|\rho_u Q^{\frac{1}{2}} \xi\|_{L^2}^2. \end{aligned}$$

This is valid for all smooth function  $\xi$  of  $L^2(\mathbb{T}^N)$  so we deduce that

$$M_t := \rho_t - \rho_0 - \int_0^t \operatorname{div}_x(\sigma(\rho_s)^{-1} K \nabla_x \rho_s) \, ds - \int_0^t \rho_s H \, ds, \quad t \in [0, T],$$

is a martingale with quadratic variation

$$\int_0^t \rho_s Q^{\frac{1}{2}} \left( \rho_s Q^{\frac{1}{2}} \right)^* \, ds.$$

Thanks to martingale representation Theorem, see [5, Theorem 8.2], up to a change of probability space, there exists a cylindrical Wiener process  $W$  such that

$$\rho_t - \rho_0 - \int_0^t \operatorname{div}_x(\sigma(\rho_s)^{-1} K \nabla_x \rho_s) \, ds - \int_0^t \rho_s H \, ds = \int_0^t \rho_s Q^{\frac{1}{2}} \, dW_s, \quad t \in [0, T].$$

This gives that  $\rho$  has the law of a weak solution to the equation (2.13) with paths in  $C([0, T], H^{-\eta}) \cap L^2(0, T; L^2)$ . Since this equation has a unique solution with paths in the space  $C([0, T], H^{-\eta}) \cap L^2(0, T; L^2)$ , and since pathwise uniqueness implies uniqueness in law, we deduce that  $P$  is the law of this solution and is uniquely determined. Finally, by the uniqueness of the limit, the whole sequence  $(P^\varepsilon)_{\varepsilon > 0}$  converges to  $P$  weakly in the spaces of probability measures on  $C([0, T], H^{-\eta})$  and  $L^2(0, T; L^2)$ . This concludes the proof of Theorem 2.2.

## References

- [1] C. Bardos, F. Golse, and B. Perthame. The Rosseland approximation for the radiative transfer equations. *Comm. Pure Appl. Math.*, 40(6):691–721, 1987.
- [2] C. Bardos, F. Golse, B. Perthame, and R. Sentis. The nonaccretive radiative transfer equations: existence of solutions and Rosseland approximation. *J. Funct. Anal.*, 77(2):434–460, 1988.
- [3] P. Billingsley. *Convergence of Probability Measures*. Wiley Series in Probability and Statistics. John Wiley & Sons, 2009.
- [4] T. Cazenave and A. Haraux. *An introduction to semilinear evolution equations*. Oxford Lecture Series in Mathematics and its Applications. Clarendon Press, 1998.
- [5] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Encyclopedia of Mathematics and Its Applications. Cambridge University Press, 2008.
- [6] A. de Bouard and M. Gazeau. A diffusion approximation theorem for a nonlinear PDE with application to random birefringent optical fibers. *Ann. Appl. Probab.*, 22(6):2460–2504, 2012.
- [7] A. Debussche and J. Vovelle. Diffusion limit for a stochastic kinetic problem. *Commun. Pure Appl. Anal.*, 11(6):2305–2326, 2012.
- [8] P. Degond, T. Goudon, and F. Poupaud. Diffusion limit for nonhomogeneous and non-micro-reversible processes. *Indiana Univ. Math. J.*, 49(3):1175–1198, 2000.

- [9] J.P. Fouque, J. Garnier, G. Papanicolaou, and K. Solna. *Wave Propagation and Time Reversal in Randomly Layered Media*. Stochastic Modelling and Applied Probability. Springer, 2010.
- [10] P.-E. Jabin. Averaging lemmas and dispersion estimates for kinetic equations. *Riv. Mat. Univ. Parma (8)*, 1:71–138, 2009.
- [11] G. C. Papanicolaou, D. Stroock, and S. R. S. Varadhan. *Martingale approach to some limit theorems*. Duke Univ. Math. Ser. Duke Univ., 1977.
- [12] J. Simon. Compact sets in the space  $L^p(0, T; B)$ . *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.